Recap

Inner products: $\mu : V \times V \longrightarrow \mathbb{F}$ $(= \mathbb{R}$ or $\mathbb{C})$ satisfying
$(\langle u, v \rangle)$

- $\forall u \quad \mu(u, \cdot) : V \longrightarrow \mathbb{F}$ is a linear transformation
- $\forall u, v \quad \mu(u, v) = \overline{\mu(v, u)}$
- $\forall v \quad \mu(v, v) \in \mathbb{R}_{\geq 0}$ with $\mu(v, v) = 0 \Leftrightarrow v = 0$

Cauchy-Bunyakovsky-Schwarz: $|\langle u, v \rangle|^2 \leq \langle u, u \rangle \cdot \langle v, v \rangle$

$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$

Orthonormal set: $\forall u \neq v \in S \quad \langle u, v \rangle = 0$

$\forall u \in S \quad \|u\| = 1$

Gram-Schmidt: Every finite dimensional inner product space has an orthonormal basis.

Fourier coefficients

Fix an orthonormal basis $B = \{w_1, \ldots, w_n\}$ for V.

- $\forall v \in V$ $\exists$ unique $c_1, \ldots, c_n$ s.t. $v = \sum c_i \cdot w_i$
  $\forall i \in [n]$ we have $c_i = \langle w_i, v \rangle$

- Parseval's identity: $\forall$ $u, v \in V$

$$\langle u, v \rangle = \sum_{i=1}^{n} \overline{\langle w_i, u \rangle} \cdot \langle w_i, v \rangle$$

Any inner-product is co-ordinate-wise product

Ex: Let $v = \sum c_i w_i$ have $\|v\| = 1$. Then $|\{i \mid |c_i| \geq \delta\}| \leq 1/\delta^2$.

Riesz representation theorem

▸ Let V be a finite dimensional inner product space over $\mathbb{F}$.
Let $\alpha : V \longrightarrow \mathbb{F}$ be a linear transformation.

Then $\exists$ unique $z \in V$ s.t. $\forall v \in V$ $\alpha(v) = \langle z, v \rangle$

(Any scalar-valued linear transformation is basically an inner product)

<u>Proof</u>: Let $B = \{w_1, \ldots, w_n\}$ be an orthonormal basis

Adjoints

Say V, W inner product spaces over same $\mathbb{F}$.

Linear $\varphi : V \longrightarrow W$

$\varphi^* : W \longrightarrow V$ is called an adjoint of φ if

$$\forall v \in V,\ w \in W \qquad \langle w, \varphi(v) \rangle_W = \langle \varphi^*(w), v \rangle_V$$

Example (Matrices)

$V = \mathbb{R}^n$, $W = \mathbb{R}^m$, standard inner product

$A \in \mathbb{R}^{m \times n}$, $\varphi_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\quad \langle w, \varphi_A(v) \rangle = \ ?$

$B \in \mathbb{R}^{n \times m}$ $\quad \varphi_B : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ $\quad \langle \varphi_B(w), v \rangle =$

Non-matrix examples

$V = C([0,1], [-1,1])$ $\qquad$ $W = C([0,\tfrac{1}{2}], [-1,1])$

$\langle f_1, f_2 \rangle = \int_0^1 f_1(x) f_2(x)\, dx$ $\qquad$ $\langle g_1, g_2 \rangle = \int_0^{1/2} g_1(y) g_2(y)\, dy$

$\varphi : V \to W$ $\qquad$ $\varphi(f)(y) = f(2y)$

$\varphi^*(g) = ?$

Ex: For $f, g \in \text{Fib}$ $\quad \langle f, g \rangle = \sum_{n=0}^{\infty} \dfrac{f(n) \cdot g(n)}{c^n}$ $\quad (c > 4)$

For $\varphi_{\text{left}} : \text{Fib} \to \text{Fib}$, find φ^*_{left}

Existence and uniqueness of adjoints

- V, W finite dimensional i.p. spaces over same $\mathbb{F}$.

  $\varphi : V \longrightarrow W$. Then $\exists$ unique adjoint $\varphi^* : W \longrightarrow V$

<u>Proof</u>: Fix a $w \in W$. $\langle \varphi^*(w), v \rangle = \langle w, \varphi(v) \rangle \;\; \forall v$

What should this be?

Self-adjoint linear transformations

- $\varphi : V \longrightarrow V$ is self-adjoint if $\varphi = \varphi^*$

  i.e. $\forall v, w \in V \quad \langle w, \varphi(v) \rangle = \langle \varphi(w), v \rangle$

Examples

- $A \in \mathbb{R}^{n \times n}$ is self-adjoint if $A = A^T$

- $A \in \mathbb{C}^{n \times n}$ is self-adjoint if $A = \overline{A}^T$

$$\boxed{\begin{matrix} BB^T \\ B^T B \end{matrix}}$$

Properties of self-adjoint transformations

Let $\varphi : V \longrightarrow V$ be a self-adjoint transformation. Then

- All eigenvalues of φ are real
- If $\varphi(w_1) = \lambda_1 w_1, \ldots, \varphi(w_k) = \lambda_k w_k$ for distinct $\lambda_1 \ldots \lambda_k$

  then $w_1, \ldots, w_k$ are orthogonal.

Proof:

$$\varphi(w) = \lambda \cdot w$$

$$\langle \varphi(w), w \rangle = \langle w, \varphi(w) \rangle$$

$$\langle \lambda \cdot w, w \rangle = \langle w, \lambda \cdot w \rangle$$

$$\bar{\lambda} \cancel{\langle w, w \rangle} = \lambda \cdot \cancel{\langle w, w \rangle}$$

$$\lambda_j \langle w_i, w_j \rangle = \langle w_i, \varphi(w_j) \rangle = \langle \varphi(w_i), w_j \rangle = \lambda_i \cdot \langle w_i, w_j \rangle$$

Real Spectral Theorem

- Let V be a finite dimensional i.p. space and let $\varphi : V \to V$ be self-adjoint. Then φ has an **orthonormal basis of eigenvectors** (orthogonally diagonalizable)

Implications

$\varphi : V \to V$ self-adjoint $\qquad w_1, \ldots, w_n$ orthonormal

$\qquad \lambda_1, \ldots, \lambda_n$

Say $v = \sum c_i \cdot w_i \qquad c_i = \langle w_i, v \rangle$

$$\varphi(v) = \sum c_i \cdot \varphi(w_i) = \sum c_i \cdot \lambda_i \cdot w_i = \sum \lambda_i \cdot \langle w_i, v \rangle \cdot w_i$$

Proof of the real spectral theorem

$\varphi : V \longrightarrow V$ $(\varphi = \varphi^*)$. Find orthonormal basis of eigenvectors

Assume Claim: Such φ has at least one eigenval.
(will prove later)

Induction on $\dim(V)$

$\dim(V) = 1$

By claim, $\exists v \in V \setminus \{0_V\}$
$\lambda \in \mathbb{F}$ s.t.

$\varphi(v) = \lambda \cdot v$

$w = \dfrac{v}{\|v\|}$

$\varphi\left(\dfrac{v}{\|v\|}\right) = \lambda \cdot \dfrac{v}{\|v\|}$

$\dim(V) = n$ (assume for $\dim(V) = n-1$)

$\exists\, w \neq 0_V$ s.t. $\varphi(w) = \lambda \cdot w$

Let $w^\perp = \{v \in V \mid \langle v, w \rangle = 0\}$

- $w^\perp$ is a subspace of V
- $\dim(w^\perp) = n-1$
- $\forall u \in w^\perp$ $\varphi(u) \in w^\perp$

$\Bigg\}$ Ex.

$\varphi : w^\perp \longrightarrow w^\perp$ has basis $\{w_1 \dots w_{n-1}\}$